Approximations to the Negative Binomial distribution

The Negative Binomial distribution $\text{NegBinomial}(\rho, s)$ models the total number of trials ($n$ trials = $s$ successes plus $n$ failures) it takes to achieve $s$ successes, where each trial has the same probability of success $\rho$.

Normal approximation to the Negative Binomial

When the number of successes $s$ required is large, and $\rho$ is neither very small nor very large, the following approximation works pretty well:

$$\text{NegBinomial}(\rho, s) \approx \text{Normal} \left( \frac{1}{\rho}, \sqrt{s(1 - \rho)/\rho^2} \right)$$

The approximation can be justified via Central Limit Theorem, because the $\text{NegBinomial}(\rho, s)$ distribution can be thought of as the sum of $s$ independent $\text{NegBinomial}(\rho, 1)$ distributions, each with mean $\frac{1}{\rho}$ and standard deviation $\sqrt{\frac{1-\rho}{\rho^2}}$.

The difficulty lies in knowing whether, for a specific problem, the values for $s$ and $\rho$ fall within the bounds for which the Normal distribution is a good approximation. The smaller the value of $\rho$, the longer the tail of a $\text{NegBinomial}(\rho, 1)$ distribution:
As $p$ gets very small, the NegBinomial($p,1$) becomes an Exponential distribution (see below), and so we can use a Gamma approximation to the NegBinomial instead of a Normal. On the other hand, as $p$ is large, so the NegBinomial($p,1$) distribution gets more skewed, so $s$ would need to be much larger for a Normal approximation (which has to overcome this skewness) to be appropriate:

\[ \text{NegBinomial}(0.5,s) \text{ distributions and their corresponding Normal distribution approximations} \]

\[ \text{NegBinomial}(0.9,s) \text{ distributions and their corresponding Normal distribution approximations, showing that when } p \text{ is large, } s \text{ needs to be higher for the Normal approximation to work well.} \]

**Gamma approximation to the Negative Binomial**

The Poisson process can be derived from the Binomial process by making $n$ extremely large while $p$ becomes very small, but within the constraint that $np$ remains finite. In a Poisson process, the Gamma($0,b,a$) distribution models the 'time' until observing $a$ events where $b$ is the mean time between events. The NegBinomial distribution is the binomial equivalent, modeling the total number of trials to achieve $s$ successes where $[(1/p)-1]$ is the mean number of failures per success. The NegBinomial in Crystal Ball includes the $s$ successes which in terms of a Poisson process are not included in the waiting time because each event is assumed to be instantaneous. To make the two approaches more comparable, we subtract the (non-random) number of successes from the NegBinomial($p,s$) distribution to obtain the number of failures only (i.e., shift the distribution $s$ to the left). The remaining distribution models the number of failures, with mean $(1/p-1)$ failures for each success. Then, we can make the following approximation:

\[ \text{NegBinomial}(p,s) - s \sim \Gamma(0,1/(p-1),s) \quad \text{when } p \neq 0 \]
Or equivalently, using the shift parameter for the Gamma distribution:

\[
\text{NegBinomial}(p, s) \sim \Gamma(s, 1/p)
\]

when \( p \to 0 \)

For \( s = 1 \), we also have the special case:

\[
\text{Geometric}(p) - 1 \sim \text{Exponential}(1/p)
\]

when \( p \to 0 \)

When the Exponential distribution is a good approximation to the "Geometric(p) - 1" (\( p < 0.05 \) is usually good, see below), the Gamma is a good approximation to the NegBinomial.