Approximations to the Negative Binomial distribution

The Negative Binomial distribution \( \text{NegBinomial}(p, s) \) models the total number of trials (\( n \) trials = \( s \) successes plus \( n-failures \)) it takes to achieve \( s \) successes, where each trial has the same probability of success \( p \).

Normal approximation to the Negative Binomial

When the number of successes \( s \) required is large, and \( p \) is neither very small nor very large, the following approximation works pretty well:

\[
\text{NegBinomial}(p, s) \approx \text{Normal} \left( \frac{s}{p}, \sqrt{s(1-p)p^2} \right)
\]

The approximation can be justified via Central Limit Theorem, because the NegBinomial\((p,s)\) distribution can be thought of as the sum of \( s \) independent NegBinomial\((p,1)\) distributions, each with mean \( \frac{1}{p} \) and standard deviation \( \sqrt{\frac{1-p}{p^2}} \).

The difficulty lies in knowing whether, for a specific problem, the values for \( s \) and \( p \) fall within the bounds for which the Normal distribution is a good approximation. The smaller the value of \( p \), the longer the tail of a NegBinomial\((p,1)\) distribution:
As \( p \) gets very small, the NegBinomial(\( p, 1 \)) becomes an Exponential distribution (see below), and so we can use a Gamma approximation to the NegBinomial instead of a Normal. On the other hand, as \( p \) is large, so the NegBinomial(\( p, 1 \)) distribution gets more skewed, so \( s \) would need to be much larger for a Normal approximation (which has to overcome this skewness) to be appropriate:

\[
\text{NegBinomial}(0.5, s) \text{ distributions and their corresponding Normal distribution approximations}
\]

\[
\text{NegBinomial}(0.9, s) \text{ distributions and their corresponding Normal distribution approximations, showing that when } p \text{ is large, } s \text{ needs to be higher for the Normal approximation to work well.}
\]

\[
\text{Gamma approximation to the Negative Binomial}
\]

The Poisson process can be derived from the Binomial process by making \( n \) extremely large while \( p \) becomes very small, but within the constraint that \( np \) remains finite. In a Poisson process, the Gamma(0, \( b, a \)) distribution models the 'time' until observing \( a \) events where \( b \) is the mean time between events. The NegBinomial distribution is the binomial equivalent, modeling the total number of trials to achieve \( s \) successes where \( [(1/p)-1] \) is the mean number of failures per success. The NegBinomial in Crystal Ball includes the \( s \) successes which in terms of a Poisson process are not included in the waiting time because each event is assumed to be instantaneous. To make the two approaches more comparable, we subtract the (non-random) number of successes from the NegBinomial(\( p, s \)) distribution to obtain the number of failures only (i.e. shift the distribution \( s \) to the left). The remaining distribution models the number of failures, with mean \((1/p-1)\) failures for each success. Then, we can make the following approximation:

\[
\text{NegBinomial}(p, s) - s \overset{\sim}{\Gamma}(0, 1/p-1, s) \quad \text{when} \quad p \gg 0
\]
Or equivalently, using the shift parameter for the Gamma distribution:

\[ \text{NegBinomial}(p, s) \sim \text{Gamma}(s, 1/p) \quad \text{when} \quad p \approx 0 \]

For \( s = 1 \), we also have the special case:

\[ \text{Geometric}(p) - 1 \sim \text{Exponential}(p/(1-p)) \quad \text{when} \quad p \approx 0 \]

When the Exponential distribution is a good approximation to the "Geometric(p) - 1" (\( p < 0.05 \) is usually good, see below), the Gamma is a good approximation to the NegBinomial.