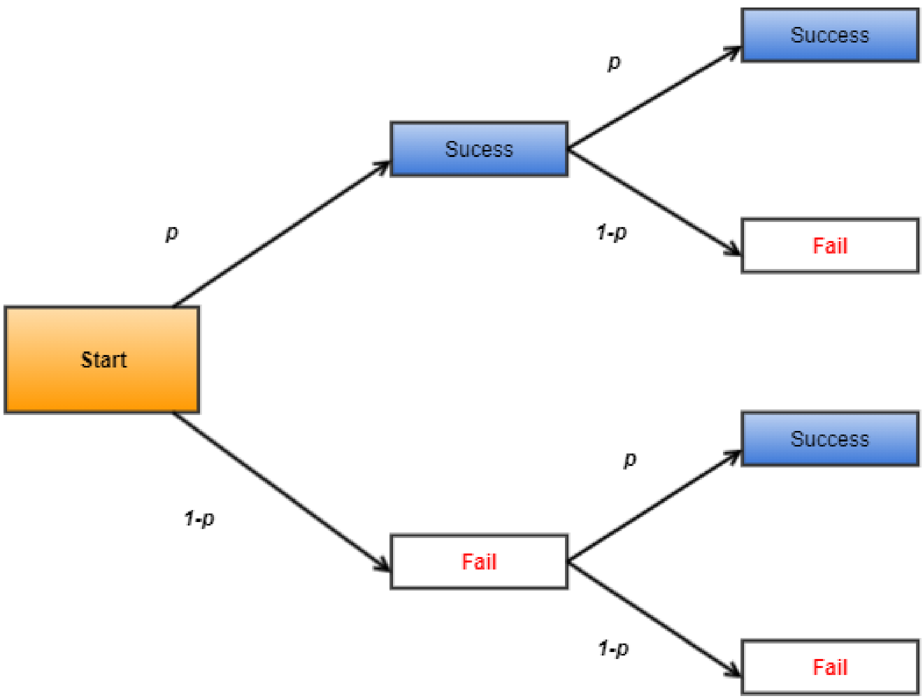
Distribution of the number of successes s in n trials, each with probability p

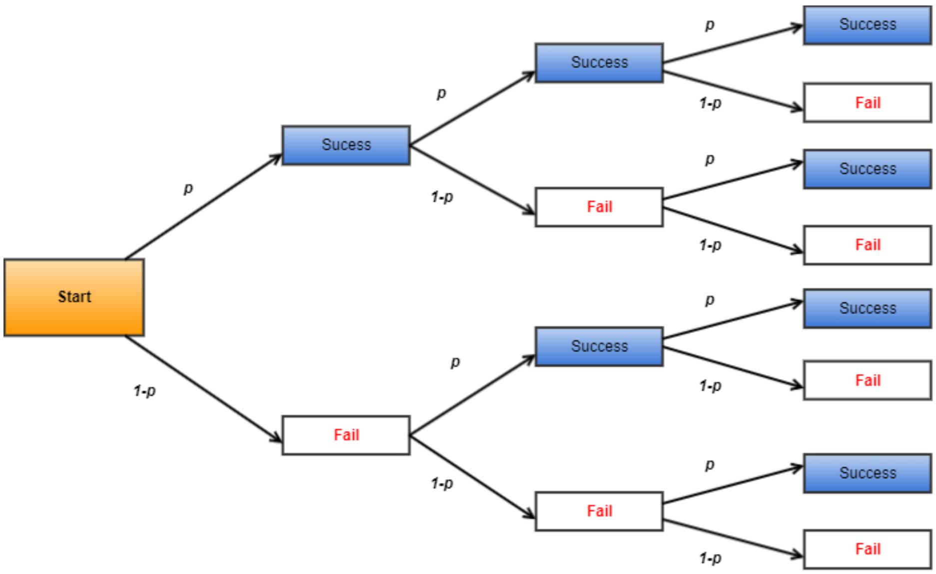
Exploring the binomial process

We start our exploration of the binomial process by looking at the probability of a certain number of successes *s* for a given number of trials *n* and probability of success *p*. Imagine we have two binomial trials, there are four possible outcomes, as shown below, namely SS, SF, FS, and FF, where SF means success followed by failure, etc.



These outcomes have probabilities p^2, $p(1-p)$, $(1-p)p$ and $(1-p)^2$ respectively. The binomial process considers each success to be identical and therefore does not differentiate between the two events SF and FS: they are both just one success in two trials. The probability of one success in two trials is then just $2p(1-p)$: the 2 in this equation is the number of different paths that result in 1 success in 2 trials.

Now imagine that we have three trials:



The eight outcomes are: SSS, SSF, SFS, SFF, FSS, FSF, FFS, and FFF. There is thus 1 event producing 3 'successes', 3 events producing 2 successes, 3 events producing 1 success and 1 event producing no successes for three trials.

The binomial distribution equation

In general, the number of ways that we can get *s* successes from *n* trials can be calculated directly using the binomial coefficient ${}_nC_s$, which is given by:

$${}_nC_s = \binom{n}{s} = \frac{n!}{s!\left(n-s\right)!}$$

We can check this is right, by choosing *n* =3, (remembering that 0! = 1) then:

$$\binom{3}{0} = \frac{3!}{0!(3)!} = 1$$

$$\binom{3}{1} = \frac{3!}{1!(2)!} = 3$$

$$\binom{3}{2} = \frac{3!}{2!(1)!} = 3$$

$$\binom{3}{3} = \frac{3!}{3!(0)!} = 1$$

which match the number of combinations we have already calculated. Each of the ways of getting *s* successes in *n* trials has the same probability, namely $p^s(1\text{-}p)^{n\text{-}s}$, so the probability of observing *x* successes in *n* trial is given by:

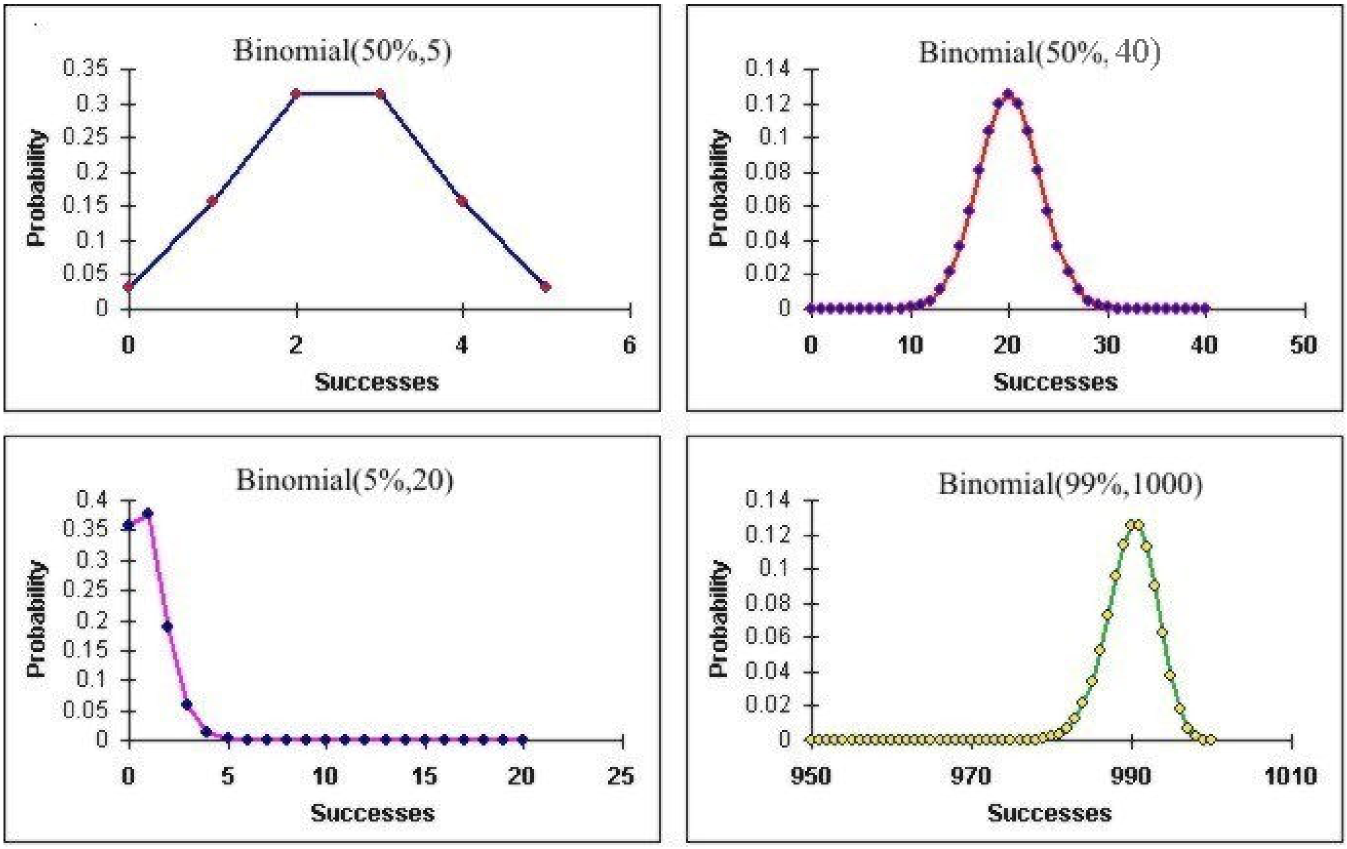
$$p_{Bin}(x) \binom{n}{x} p^x (1-p)^{n-x}$$

which is the probability mass function of the Binomial(*p,n*) distribution. In other words, the number of successes *s* one will observe in *n* trials, where each trial has the same probability of success is given by:

s = Binomial(*p,n*)

Examples of the Binomial

The four figures below show the distribution for a number of different p's (probabilities) and n's (trials).



The first figure above shows the number of tails (here called success (s)) if we toss a coin five times could be 0, 1, 2, 3, 4 or 5 with the probabilities displayed on the y-axes. The figure to the right shows the same situation, but now with 40 tosses. The most likely outcome is 20 tails , but it could be as low as 10 or as high as 30. Note that this figure looks somewhat like a bell-shaped Normal distribution.