Time to wait to observe alpha events

Motivation

If you have understood how the Poisson process works and are willing to accept that a Gamma distribution models the time to wait to observe events, this section is superfluous to your needs. We explain here:

- How the Exponential distribution models the time to wait for the first event and arises naturally out of a memoryless system;
- And therefore why the distribution of the time to wait to observe an event remains the same even if one has waited a while;
- How the Gamma distribution is the sum of a number of Exponential distributions, and thus is the waiting time distribution for events.

Deriving the Exponential distribution

The Poisson process assumes that there is a constant probability that an event will occur per increment of time. If we consider a small element of time *Dt*, then the probability an event will occur in that element of time is *kDt*, where *k* is some constant. Now let *P(t)* be the probability that the event will not have occurred by time *t*. The probability that an event occurs the first time during the small interval *Dt* after time *t* is then *kDtP(t)*. This is also equal to *P(t) - P(t+D t)* and we have:

$$\left[\frac{P\left(t+\Delta t\right)-P\left(t\right)}{P\left(t\right)}\right]=-k\Delta t$$

Making *Dt* infinitesimally small, this becomes the differential equation:

$$\frac{dP(t)}{P(t)}=-kdt$$

Integration gives:

$$In[P(t)]=-kt$$

$$P(t)=exp[-kt]$$

If we define *F(t)* as the probability that the event will have occurred before time *t* (i.e. (1-*P(t)*), the cumulative distribution function for *t*), we then have:

$$F(t)=1-exp[-kt]$$

which is the cumulative distribution function for an Exponential distribution Exponential(*k*) with mean 1/*k*. Thus 1/*k* is the mean time between occurrences of events or, equivalently, *k* is the mean number of events per unit time, which is the Poisson parameter . The parameter 1/, the mean time between occurrences of events, is given the notation .

Derivation of the Gamma distribution

We have shown above that the time until occurrence of the first event for a Poisson distribution is given by:

t_1 = Exponential(1/)   where = 1/

From the mathematics of convolutions we have:

$$Z = X + Y$$
$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_Y(z-x) dx$$

For X = Gamma(0,,) and Y = Exponential(1/) this gives:

$$f_z(z) = \int_0^z \frac{t^{\alpha-1}}{\beta^\alpha(\alpha-1)!} e^{-\frac{t}{\beta}} \frac{1}{\beta} e^{\frac{(z-t)}{\beta}} dt$$
$$= \frac{1}{\beta^{\alpha+1}(\alpha-1)!} e^{-\frac{z}{\beta}} \int_0^z t^{\alpha-1} dt \frac{t^\alpha}{\beta^{\alpha+1}\alpha!} e^{-\frac{z}{\beta}}$$

This is equal to a Gamma (0,, +1).

Since Gamma(0,,1) = Y = Exponential(1/) we have proven by induction that:

$$Gamma(0, \beta, a) = \sum_a Exponential(1/\beta)$$

The memoryless property of an Exponential distribution

The probability that the first event will occur at time *x*, given it has not yet occurred by time *t* (*x*>*t*), is given by:

$$f(x \mid x > t) = \frac{f(x)}{1 - F(t)} = \frac{1}{\beta} exp\left[-\frac{(x-t)}{\beta}\right]$$

which is another Exponential distribution. Thus, although the event may not have occurred after time *t*, the remaining time until it will occur has the same probability distribution as it had at any prior point in time.